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TIME-OPTIMAL CONTROL SYNTHESIS FOR A FOURTH-ORDER NONLINEAR SYSTEM

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Iu. I. BERDYSHEV
(Sverdlovsk)

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On the basis of Pontriagin's maximum principle we establish the structure of the optimal control and of the optimal trajectories, using the properties of the system being analyzed. We propose a rule for the construction of the program control satisfying the maximum principle. In the case when the terminal state lies outside some bounded region we prove that the rule mentioned determines the optimal control and permits us to solve the synthesis problem.

1. Statement of the problem. Let the motion of a point in the xy-plane be described by the system of equations

$$x^{\cdot} = v\cos\varphi, \quad y^{\cdot} = v\sin\varphi, \quad \varphi^{\cdot} = \frac{K_1}{v}u_1, \quad v^{\cdot} = K_2u_2 \tag{1.1}$$

where $\varphi = \varphi(t)$ is the angle between the x-axis and the direction of the velocity vector $\mathbf{v}(t) = (x^{\cdot}, y^{\cdot})$, K_1 and K_2 are given positive constants, while $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are measurable control functions satisfying the conditions

$$|u_1| \leqslant 1, \quad |u_2| \leqslant 1 \tag{1.2}$$

In this paper we solve the problem on the fastest hitting of the representative point (x, y, φ, v) onto the manifold $(x = 0, y = 0)$ from a given initial state $(x_0, y_0, \varphi_0, v_0)$. We assume that $v_{-10} > c$, where $c = \text{const} > 0$. Under this condition and with $K_2K_1 \leqslant 0{,}1$ we can prove the existence of a positive number $a = a(c)$ such that along a trajectory which appears to be optimum, we have $v(t) > a$. Then, the optimal control existence theorem is valid [1].

By a trajectory of system (1.1), (1.2) it is expedient to understand the projection of the phase trajectory onto the xy-plane and carry out the investigation in this plane. The origin $O = (0, 0)$ on this plane corresponds to the manifold $(x = 0, y = 0)$. In the case given the Hamilton function H has the form [1]

$$H = \psi_1 v \cos\varphi + \psi_2 v \sin\varphi + \psi_3 \frac{1}{v} K_1 u_1 + \psi_4 K_2 u_2 \tag{1.3}$$

where the auxiliary functions $\psi_1, \psi_2, \psi_3, \psi_4$ satisfy the system of Eqs. (1.4) and the transversality conditions (1.5)

$$\psi_1^{\cdot} = 0, \ \psi_2^{\cdot} = 0, \ \psi_3^{\cdot} = \psi_1 v \sin\varphi - \psi_2 v \cos\varphi \tag{1.4}$$

$$\psi_4^{\cdot} = -\psi_1 \cos\varphi - \psi_2 \sin\varphi + \psi_3 v^{-2} K_1 u_1$$

$$\psi_3(T) = \psi_4(T) = 0 \tag{1.5}$$

(T is the instant of hitting onto the origin). Using the first two equations of system (1.1), we have

$$\psi_1 = c_1, \quad \psi_2 = c_2, \quad \psi_3^{\cdot} = c_1 y^{\cdot} - c_2 x^{\cdot} \tag{1.6}$$

Hence, allowing for the boundary conditions (1.5) and $x(T) = y(T) = 0$, we have $\psi_3 = c_1 y - c_2 x$. In the xy-plane

$$c_1 y - c_2 x = 0 \tag{1.7}$$

is the equation of a straight line which we call the switching line. We subsequently consider the coefficients of this straight line as normalized: $c_1 = \cos\theta$ and $c_2 = \sin\theta$ (θ is the angle between the x-axis and the vector (c_1, c_2)). From (1.4) and (1.3) it follows that the equation for $\psi_4(t)$ can be reduced to the form

$$\psi_4^{\cdot} = -v^{-1} K_2 \psi_4 \cdot u_2 + v^{-1} [H - 2v \cos(\varphi - \theta)] \tag{1.8}$$

According to the maximum principle the optimal control satisfies the relations

$$u_1 = \operatorname{sign} \psi_3, \ \ \psi_3 \neq 0, \ \ u_2 = \operatorname{sign} \psi_4, \ \ \psi_4 \neq 0 \tag{1.9}$$

We note that certain aspects of the optimality of the motion of system (1.1) were touched upon in [2]. The problem being examined was solved in [3] for $v = \text{const}$ and in a somewhat different formulation, in [4].

2. Certain properties of system (1.1), (1.4). 1°. Let $\psi_3(t) = 0$, $t \in [t_a, t_b]$, $t_a < t_b$, i.e. the motion takes place along the straight line (1.7). Then from (1.1) it follows that $u_1(t) = 0$, $t \in [t_a, t_b]$.

2°. If at point $(x(t_a), y(t_a))$ the velocity vector $\mathbf{v}(t_a)$ is directed toward the origin, then the optimal trajectory for $t \geqslant t_a$ is a straight line joining this point with the origin, while the optimal control is $(u_1^\circ = 0,\ u_2^\circ = 1)$.

3°. The equalities

$$Hv - 2K_2u_2(c_1x + c_2y) - vK_2\psi_4u_2 = \text{const} \tag{2.1}$$

$$K_1u_1\psi_3 - 2K_2u_2\ (c_1x + c_2y) + v^2\cos\ (\varphi - \theta) = \text{const} \tag{2.2}$$

hold on the intervals of constancy of the control functions u_1 and u_2 and of the Hamilton function H

Suppose that at some instant t_2 the function $\psi_4(t_2) = \psi_4(T) = 0,\ \psi_4(t) > 0,\ t \in (t_2,\ T)$, while at an instant t_3 the function $\psi_4\ (t_3) = 0,\ \psi_4(t) < 0,\ t \in (t_3,\ t_2)$; then from (2.1) and (1.5) follow the relations

$$v\ (t)\ \ K_2\psi_4\ (t) = H\ (v\ (t) - v\ (T)) - 2K_2\ (c_1x\ (t) + c_2y\ (t)) \tag{2.3}$$

$$t \in (t_2,\ T]$$

$$c_1x\ (t_2) + c_2y\ (t_2) < 0$$

$$v\ (t)K_2\ |\psi_4\ (t)| = H\ (v\ (t) - v\ (t_2 - 0)) + 2K_2\ (c_1x + c_2y)|^t{}_{t_2-0},$$

$$t \in (t_3,\ t_2)$$

4°. Let the control functions u_1 and u_2 be constant on the interval $[t_\alpha,\ t_\beta]$, taking the values ± 1. Then

$$y - y\ (t_\alpha) = b\ [v^2K_2^{-1}u_2\ \ \ (2\sin\varphi - K^{-1}u_1u_2\ \ \cos\varphi)]|_{t_\alpha}{}^t \tag{2.6}$$

$$x - x\ (t_\alpha) = b\ [v^2K_2^{-1}u_2\ \ \ (2\cos\varphi + K^{-1}u_1u_2\ \ \sin\varphi)]|_{t_\alpha}{}^t$$

$$v\ (t) = v\ (t_\alpha)\exp\ (Ku_1u_2\ (\varphi\ (t) - \varphi\ (t_\alpha))$$

$$v\ (t) = v\ (t_\alpha) + K_2u_2\ (t - t_\alpha), \qquad t \in [t_\alpha,\ t_\beta]$$

$$K = K_2K_1^{-1}, \qquad b = (4 + K^{-1})^{-1}$$

Hence we see that the corresponding trajectories are logarithmic spirals.

5°. Let us consider the $v\varphi$-plane to each point of which corresponds a radius-vector of length v turned through an angle φ relative to some fixed z-axis. On the $v\varphi$-plane the relations

$$v\cos\ (\varphi - \theta) = H, \qquad 2v\cos\ (\varphi - \theta) = H \tag{2.7}$$

are the equations of straight lines orthogonal to the straight line $\varphi = \theta$. From (1.3) and (1.5) it follows that the point $(v\ (T),\ \varphi\ (T))$ is on the first straight line in (2.7). The second straight line divides the $v\varphi$-plane into two halfplanes Π_1, Π_2. Let the quantity $H - 2v\cos\ (\varphi - \theta)$ be negative in halfplane Π_1 and positive in Π_2

Lemma 2.1. As the representative point moves in the halfplane Π_1 the function $\psi_4\ (t)$ decreases, while in Π_2 it can change sign only from minus to plus.

The validity of the Lemma follows immediately from (1.8).

3. Necessary conditions for the optimality of the trajectory. Structure of the optimal control. The sign of the expression $\sigma_0 = x_0\sin\varphi_0 - y_0\cos\varphi_0$ determines the relative positions of the origin and the straight line

$$(x - x_0)\sin\varphi_0 = (y - y_0)\ \cos\varphi_0 \tag{3.1}$$

From the inequality $\sigma_0 > 0$ $(\sigma_0 < 0)$ it follows that the origin is to the left (to the right) if we view from point $M_0 = (x_0, y_0)$ along the direction of vector $\mathbf{v}_0$.

Let $(u_1^\circ(t), \ u_2^\circ(t))$ be the optimal control and $M_1 = (x(t_1), \ y(t_1))$ be the first point of contact of the trajectory with the switching line. Using property 2° and the transversality condition (1.5), we can prove

Lemma 3.1. Let $\sigma_0 \neq 0, \ u_1^\circ(t) = \operatorname{sign} \sigma_0, \ t \in [0, t_1)$, then (a) on the arc M_0M_1 there is no point other than M_1, at which the velocity vector is directed toward the origin, (b) at point M_1 the velocity vector is directed toward the origin and $u_1^\circ(t) = 0, \ t \in [t_1, \ T]$.

Lemma 3.2. Under the hypotheses of Lemma 3.1 the optimal control function $u_2^\circ(t)$ does not switch more than once and the sequence of values of $u_2^\circ(t)$ can only be one of the following: (-1), (1), $(-1, \ 1)$.

The proof, which we omit, relies on Lemma 2.1 and on relations (2.4) and (2.5).

Using Lemmas 3.1 and 3.2 we can show the validity of the following theorems.

Theorem 3.1. At no point, other than the initial one of the optimum trajectory, is the velocity vector directed away from the coordinate origin. The optimum control function $u_1^\circ(t)$ does not switch more than twice, and the sequence of values of $u_1^\circ(t)$ can only be one of the following: for $\sigma_0 \neq 0$ ($\operatorname{sign} \sigma_0$), either ($\operatorname{sign} \sigma_0, \ 0$), ($-\operatorname{sign} \sigma_0, \ \operatorname{sign} \sigma_0$), or ($-\operatorname{sign} \sigma_0, \operatorname{sign} \sigma_0, \ 0$), and for $\sigma_0 = 0$ (0), either(± 1), or ($\pm$ 1,0).

Theorem 3.2. The optimum control function $u_2^\circ(t)$ has no more than one switching, while the sequence of values of $u_2^\circ(t)$ can only be one of the following: (-1), $(+1)$, $(-1, \ +1)$.

4. Certain properties of the function $\psi_4(t)$. Let the control $u_1(t) = u_2(t) = 1, \ t \in [t_\alpha, \ T]$ satisfy the maximum principle. Then, having substituted (1.9) and (2.6) into (2.1), we have

$$K_2\psi_4(t) - K_2\psi_4(T)\,E = v(t)\,\Phi(\varphi^*(t), \alpha), \quad t \in [t_\alpha, \ T] \tag{4.1}$$

$$\Phi(\varphi^*(t), \alpha) = E(1 - E)\cos\alpha + 2b\{(2\cos\alpha - K^{-1}\sin\alpha)E - (2\cos(\varphi^*(t) + \alpha) - K^{-1}\sin(\varphi^*(t) + \alpha)\}$$

$$\varphi^*(t) = \varphi(T) - \varphi(t), \qquad E = \exp(K\varphi^*(t)), \qquad \alpha = \theta - \varphi(T)$$

If $t_\alpha = t_2$, and $\psi_4(T) = 0$ (t_2 is the switching instant for u_2), then

$$\Phi(\varphi^*(t_2), \ \alpha) = 0 \tag{4.2}$$

Definition. Let $\sigma_0 > 0$. We say that the control $(u_1(t), \ u_2(t)), \ t \in [0, \ T]$ belongs to class X if (a) this control satisfies the maximum principle and the transversality condition, (b) on the trajectory corresponding to this control there is no point other than the initial point, at which the velocity vector is directed away from the point $(x(T), \ y(T))$, (c) the control functions u_1 and u_2 take during the motion one of the following sequences of values:

$$u_1: (0), \ (1), \ (1,0), \ (-1, \ 1, \ 0); \quad u_2: (-1), \ (1), \ (-1, \ 1)$$

If a control $(u_1, \ u_2)$ belongs to class X and $u_1(t) = -1, \ t \in [0, \ t_1)$, then

$$u_2(t) = -1, \qquad t \in [0, \ t_1] \tag{4.3}$$

Using Lemma 2.1, the $v\varphi$-plane, and the form of the derivatives $\psi_4^{\cdot}(t)$ and $\psi_4^{\cdot\cdot}(t)$, we can prove the validity of

Lemma 4.1. Let the control

$$u_1(t)=\begin{cases}+1, & t\in[t_2,\tau),\\ 0, & t\in[\tau,T],\end{cases}\quad u_2(t)=1,\quad t\in[t_2,T]$$

belong to class X and let $\psi_4(t_2)=0$, then on the interval $[t_2, T]$ the function $\psi_4(t)$ increases strictly at first and then strictly decreases.

5. Construction of a control from class X. Using symmetry arguments it suffices to solve the problem for $\sigma_0\geqslant 0$. The trajectory with control ($u_1=\pm 1$, $u_2=1$) is called an acceleration trajectory, while with control ($u_1=\pm 1$, $u_2=-1$), a deceleration trajectory. We reckon the controls from class X as right-continuous.

Let us show that an instant $\tau^*>0$ exists such that for any $\tau\leqslant\tau^*$ the control

$$u_1(t)=u_2(t)=1,\quad t\in[0,\tau] \tag{5.1}$$

leading the representative point to the position $(x(\tau), y(\tau))$ satisfies the maximum principle. To do this we take the straight line

$$(y-y(\tau))c_1-(x-x(\tau))c_2=0,\quad c_1=\cos\varphi(\tau) \tag{5.2}$$
$$c_2=\sin\varphi(\tau)$$

as the switching line. By $\psi_{41}(\tau, t)$ we denote the function $\psi_4(t)$ computed for the switching line (5.2) and under the condition $\psi_{41}(\tau,\tau)=0$. For small τ, from (4.1) we have $\psi_{41}(\tau,t)>0$, $t\in[0,\tau)$. Hence it follows that control (5.1) satisfies the maximum principle for small τ and for the function $\psi_4(t)=\psi_{41}(\tau,t)$. We increase the instant τ and determine the value of $\psi_4(\tau,0)$. Let $\tau=\tau^*$ be the smallest instant for which $\psi_{41}(\tau^*,0)=0$. Relations (1.9) and (1.5) hold for any $\tau\leqslant\tau^*$ and for the switching line (5.2); consequently, control (5.1) satisfies the maximum principle.

We shall show that control (5.1) does not satisfy the maximum principle for $\tau>\tau^*$ and for the line (5.2). We assume the contrary. Then an instant $\tau_1>0$ exists such that $\varphi(\tau)-\varphi(\tau_1)=\varphi(\tau^*)-\varphi_0$. In view of Lemma 4.1 and of equality (4.2) we have $\psi_{41}(\tau,t)>0$, $t\in(\tau_1,\tau)$, $\psi_{41}(\tau,\tau_1-0)<0$. Therefore, relations (1.9) and (1.5) cannot be fulfilled simultaneously when $\tau>\tau^*$. Q.E.D.

We make the following construction. We move a certain time $\tau>\tau^*$ along an acceleration trajectory, and next a certain time $T_1-\tau$ along line (5.2). From (1.4) we have $\psi_4(T_1)=\psi_4(\tau)-T_1+\tau$. By choosing a sufficiently large interval $[\tau, T_1]$ we can fulfill the inequality $\psi_4(t)>0$, $t\in[0,T_1)$, under the condition $\psi_4(T_1)=0$. From Lemma 4.1 it follows that in the case being examined we can find $T_{1*}(\tau)=T_1$, namely, the smallest instant for which $\psi_4(0)=\psi_4(T_{1*})=0$, $\psi_4(t)>0$, $t\in(0,T_{1*})$. We mark the point $M_\tau=(x(T_{1*}(\tau)), y(T_{1*}(\tau)))$. When $\tau\leqslant\tau^*$ we set $T_{1*}(\tau)=\tau$. By varying τ from zero to τ_0 (τ_0 is the instant of intersection of the acceleration trajectory with the straight line (3.1)), we obtain a set of points $\{M_\tau\}$ which is a certain curve γ_{10} on the half-plane being examined. Obviously, we can define curve γ_{10} as follows. By $\psi_4(\tau,t)$ we denote the function $\psi_4(t)$ computed for the switching line (5.2) and under the initial condition $\psi_4(\tau, 0)=0$. We compute the value of $\psi_4(\tau,\tau)$. Let $\psi_4(\tau,\tau)>0$. Then, moving along the switching line with the control $(u_1 = 0,\ u_2 = 1)$, we determine the instant $T_{1*}(\tau)$ at which $\psi_4(\tau,\ T_{1*}(\tau)) = 0$. We mark the point $(x(T_{1*}(\tau)),\ y(T_{1*}(\tau)))$. If $\psi_4(\tau,\ \tau) \leqslant 0$, we set $T_{1*}(\tau) = \tau$. The set of points $\{(x(T_{1*}(\tau)),\ y(T_{1*}(\tau)))\}$ is a curve γ_{10} on the halfplane being examined, which can be written in the parametric form

$$y = y(\tau) + d(\tau)\sin\varphi(\tau), \quad x = x(\tau) + d(\tau)\cos\varphi(\tau) \tag{5.3}$$
$$d(\tau) = 2^{-1}K_2(T_{1*}(\tau) - \tau)^2 + v(\tau)(T_{1*}(\tau) - \tau)$$
$$T_{1*}(\tau) - \tau = \begin{cases} K_2^{-1}(H(\tau) - v(\tau)), & H(\tau) > v(\tau) \\ 0, & H(\tau) \leqslant v(\tau) \end{cases}$$
$$H(\tau) = v_0\cos(\varphi - \varphi(\tau)) + K_1 v_0^{-1}\,|(y_0 - y(\tau))\cos\varphi(\tau) - (x_0 - x(\tau))\sin\varphi(\tau)\,|$$

Up to the point $M^* = (x(\tau^*),\ y(\tau^*))$ the curve γ_{10} coincides with the acceleration trajectory, and then diverges from it. Using Lemma 4.1 and (4.1) we can prove that if $\tau_2 > \tau_1 > \tau^*$, then $d(\tau_2) > d(\tau_1)$. The curve γ_{10} divides the half-plane into two parts Γ_{10} ($\gamma_{10} \subset \Gamma_{10}$) and Γ_{20} (Fig. 1).

Lemma 5.1. If the origin is located in region Γ_{10}, then the control

$$u_1(t) = \begin{cases} +1, & t \in [0,\ \tau), \\ 0, & t \in [\tau,\ T], \end{cases} \quad u_2(t) = 1, \quad t \in [0,\ T] \tag{5.4}$$

(where τ is the first instant at which the velocity vector is directed toward the origin) belongs to class X.

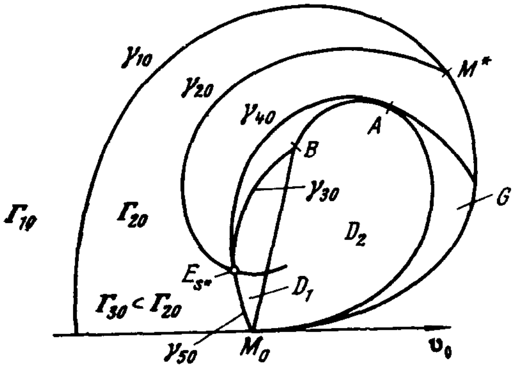

Fig. 1

Proof. Since on the interval $[0, \tau]$ the trajectory is an arc of an untwining spiral, the trajectory lies on one side of line (5.2), i.e. the first relation in (1.9) is fulfilled, while on the trajectory there is no point other than the initial point, at which the velocity vector is directed away from the origin. To fulfill the second relation in (1.9) and the transversality condition (1.5) it is sufficient to choose the function $\psi_4(t) = \psi_{41}(\tau,\ t) - \psi_{41}(\tau,\ T)$, when $\tau \leqslant \tau^*$ and the function $\psi_4(t) = \psi_4(\tau,\ t) - \psi_4(\tau,\ T)$ when $\tau > \tau^*$.

It turns out that control (5.1) does not satisfy the maximum principle for any $\tau > \tau^*$. The validity of the next statement follows from this and from the construction of curve γ_{10}

Lemma 5.2. If control $(u_1,\ u_2)$ belongs to class X, the point $(x(T),\ y(T))$ is located in region Γ_{20}, and $u_1(0) = 1$, then $u_2(0) = -1$.

From point M_0 we issue a trajectory with control $u_1(t) = 1,\ u_2(t) = -1,\ t \in [0,\ s]$, in the direction of v_0. On this trajectory we denote by A the first point at which the normal to the trajectory passes through M_0, and by B the first point at which the tangent to the trajectory passes through M_0. Let the instant $t = s_+$ correspond to point A and the instant $t = s_-$ to point B. For the point $M_s = (x(s),\ y(s))$ and for the vector v (s), taken as the initial ones, we construct the curve γ_{1s} analogous to curve γ_{10} and coinciding with it for $s = 0$. Varying $0 \leqslant s \leqslant s_-$, we obtain a set of curves γ_{1s}. On each of these curves we single out the point $(x(\tau_s^*),\ y(\tau_s^*))$ at which the curve γ_{1s} diverges from the acceleration trajectory issued from point M_s. These points form a curve γ_{20} (Fig. 1). From (4.2) we have

$$\varphi(\tau_s^*) - \varphi(s) = \mathrm{const} \tag{5.5}$$

We draw a curve γ_{30}, being the collection of first points E_s on curves γ_{1s}, at which the velocity vector is directed toward M_0. It is easy to see that the points E_s can exist only for $s \in [s_+, s_-]$; the point E_{s_-} coincides with B and the curve γ_{30} passes through M_0. In view of identity (5.5) the curves γ_{20} and γ_{30} can intersect at a unique point which we denote E_{s^*}. The curve, consisting of the arcs $M^*E_{s^*} \subset \gamma_{20}$ and $E_{s^*}M_0 \subset \gamma_{30}$ cuts out a region Γ_{30} from Γ_{20} (Fig. 1). If point E_{s^*} does not exist, then curve γ_{20} separates region Γ_{30} from Γ_{20}. By virtue of the above argumentation related to the construction of region Γ_{30}, the following lemma is valid.

Lemma 5.3. If the origin is located in region Γ_{30}, the control

$$u_1(t) = \begin{cases} 1, & t \in [0, \tau), \\ 0, & t \in [\tau, T], \end{cases} \quad u_2(t) = \begin{cases} -1, & t \in [0, s] \\ +1, & t \in [s, T] \end{cases} \tag{5.6}$$

(where τ is the first instant at which the velocity vector is directed toward the origin, s $(s < \tau)$ is the first instant at which the curve γ_{1s} passes through the origin) belongs to the class X.

By D_1 we denote the region bounded by arcs of the curves γ_{20}, γ_{30} and by the segment BM_0 (Fig. 1), by D_2 the open region bounded by the deceleration arc M_0B and the segment BM_0, while $\Gamma_{40} = \Gamma_{20} \setminus (\Gamma_{30} \cup D_1 \cup D_2)$. We choose some trajectory $L_{s\tau}$ consisting of a deceleration arc M_0M_s : $\{(x(t),\ y(t)),\ t \in [0, s]\}$ and an acceleration arc M_sO : $\{(x(t),\ y(t)),\ t \in [s, \tau],\ \tau < \tau_s^*\}$. Obviously, a trajectory $L_{s\tau}$ can hit any point of the region Γ_{40} It can be shown that for any s and $\tau \in [s, \tau_s^*)$ for which $(x(\tau),\ y(\tau)) \in \Gamma_{40}$ we can, by choosing angle α, make the function $\psi_4(t)$, computed for $c_1 = \cos(\varphi(\tau) + \alpha)$ and $c_2 = \sin(\varphi(\tau) + \alpha)$, $\psi_4(\tau) = 0$, satisfy the conditions: $\psi_4(s) = 0, \psi_4(t) > 0,\ t \in (s, \tau)$; $\psi_4(t) < 0,\ t \in [0, s)$. Angle α is uniquely determined from Eq. (4.2) with $\varphi^* = \varphi(\tau) - \varphi(s)$. If the trajectory $L_{s\tau}$ lies entirely on one side of the switch straight line defined by the angle $\theta = \varphi(\tau) + \alpha$, we relate point $(x(\tau),\ y(\tau))$ to set S_2, otherwise, point $(x(\tau),\ y(\tau))$ is referred to set S_1. Since $\tau < \tau_s^*$, it is impossible to hit the origin $0 \in S_1$ by a control from class X, which would satisfy the condition $u_1(0) = 1$. When $0 \in D_1 \cup D_2$ control (5.6) does not satisfy the maximum principle as well because the trajectory cannot lie on one side of the switch line. By Γ_{50} we denote the set $S_1 \cup D_1 \cup D_2$. It can be shown that Γ_{50} is a simply-connected open region. The deceleration arc M_0A and the arc $M_0E_{s^*} = \gamma_{50} \subset \gamma_{30}$ form a part of the boundary of region Γ_{50}. The remaining part of the boundary (we denote it γ_{40}) jointly with arc γ_{50} possesses the property that when $0 \in \gamma_{50} \cup \gamma_{40}$ the switch line defined by the angle $\varphi(\tau) + \alpha$ passes through M_0. Let $\Gamma_{60} = \Gamma_{20} \setminus \Gamma_{50}$. By virtue of the argumentation related to the construction of regions Γ_{60} and Γ_{50} and by the condition (4.3), the following lemmas are valid.

Lemma 5.4. If the origin is in the region Γ_{60}, the control (5.6) belongs to class X.

Lemma 5.5. If the origin belongs to the region Γ_{50}, the optimal control satisfies the conditions $u_1^\circ(0) = -1$, $u_2^\circ(0) = -1$.

From Lemma 5.5 it follows that when $O \in \Gamma_{50}$

$$u_1^\circ(t) = u_2^\circ(t) = -1, \quad t \in [0, t_1) \tag{5.7}$$

where t_1 is the first instant at which the curve $(\gamma_{5t_1} \cup \gamma_{4t_1})$, i.e. the boundary of the region Γ_{5t_1}, constructed for point $(x(t_1), y(t_1))$ taken as the initial point, passes through the coordinate origin. Region Γ_{5t_1} is constructed analogously to region Γ_{50}.

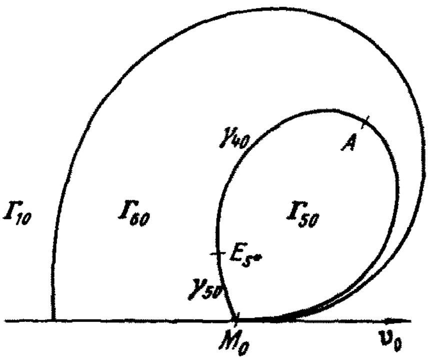

Fig. 2

Since $u_2(t_1) = -1$ (see (4.3)), $O \in \Gamma_{6t_1}$. In view of Lemma 5.4 the control

$$u_1(t) = \begin{cases} -1, & t \in [t_1, \tau) \\ 0, & t \in [\tau, T] \end{cases} \tag{5.8}$$

$$u_2(t) = \begin{cases} -1, & t \in [t_1, s) \\ +1, & t \in [s, T] \end{cases}$$

(where s is the first instant at which the curve γ_{1s}, constructed for the point $(x(s), y(s))$ taken as the initial point, passes through the origin) belongs to class X. Since $O \in \gamma_{5t_1} \cup \gamma_{4t_1}$, the switching line passes through the point $(x(t_1), y(t_1))$. Hence the first relation in (1.9) is fulfilled on the whole interval $[0, T]$. The second relation in (1.9) is fulfilled by virtue of equality (4.3). Thus when $O \in \Gamma_{50}$ the control (5.7), (5.8) belongs to class X.

6. Rule 6.1. At the initial instant we draw the straight line (3.1) and that half-plane in which the origin lies and we separate it into the regions $\Gamma_{10}, \Gamma_{50}, \Gamma_{60}$. We choose the control: (5.4) if $O \in \Gamma_{10}$; (5.6) if $O \in \Gamma_{60}$; (5.7), (5.8) if $O \in \Gamma_{50}$ (Fig. 2). From the preceding Section it follows that the control constructed by this rule belongs to class X.

We make the following construction. We move under the control $u_1(t) = u_2(t) = -1$, $t \in [0, s_3]$, where s_3 is determined from the equation $\varphi(s_3) = \varphi_0 - \pi$, and for each point $(x(t), y(t))$ taken as the initial point, we construct the region Γ_{5t}, $t \in [0, s_3]$. On the plane $\sigma_0 > 0$ these regions cover a certain region G_1. Let $G = \Gamma_{20} \cap G_1$.

Theorem 6.1. When $O \in G$ the rule 6.1 determines the optimum control.

Proof. By (u_{1*}, u_{2*}) we denote the control determined by the rule 6.1. In what follows we use an asterisk in the notation of all quantities relating to the control (u_{1*}, u_{2*}) and a degree sign in those relating to the optimum control (u_1°, u_2°). Let $O \in \Gamma_{10}$ and $u_1^\circ(0) = 1$. Then, according to Theorem 3.1, the function $u_1^\circ(t)$ has the form (5.4). We assume that $u_2^\circ(0) = -1$. Since $O \in \Gamma_{10}$, u_2 necessarily switches at some instant t_2. From (2.3) and (2.5) we have the equalities

$$v_0 K_2 \,|\psi_4^\circ(0)| = H^\circ(v_0 - 2v(t_2) + v(T^\circ)) + 2K_2(c_1^\circ x_0 + c_2^\circ y_0) \tag{6.1}$$

$$v_0 K_2 \,|\psi_{4*}(0)| = H_*(v_0 - v(T_*)) - 2K_2(c_{1*} x_0 + c_{2*} y_0)$$

We can show that

$$c_1^\circ x_0 + c_2 y_0 < c_{1*} x_0 + c_{2*} y_0 < 0, \quad H_* = v(T_*) > H^\circ = v(T^\circ) \quad (6.2)$$

In addition, from (1.1) we have $v(T_*) - v_0 = K_2 T_*$, $v(T^\circ) - 2v(t_2) + v_0 = K_2 T^\circ$. Now, using inequalities (6.2), from (6.1) we obtain $T^\circ > T_*$, which is impossible. Thus, if $O \in \Gamma_{10}$ and $u_1^\circ(0) = 1$, the control (5.4) is optimal.

Let $O \in \Gamma_{60}$ and $u_1^\circ(0) = 1$. From Theorems 3.1 and 3.2 and Lemma 5.2 it follows that $u_2^\circ(t) = -1$ until the curve γ_{1s} passes through the origin. For the subsequent motion, as was shown above, $u_2^\circ(t) = 1$. Thus, if $O \in \Gamma_{60}$ and $u_1^\circ(0) = 1$, the control (5.6) is optimal.

We shall show that $u_1^\circ(0) = 1$ if $O \in \Gamma_{10} \cup \Gamma_{60}$. We assume the contrary, i.e. $u_1^\circ(0) = -1$. From Theorem 3.1 and equality (4.3) it follows that t° a switching instant for u_1°, exists such that $u_2^\circ(0) = -1$ for $t \in [0, t^\circ]$ and $O \in (\gamma_{4t^\circ} \cup \gamma_{5t^\circ}) \subset G_1$. Consequently, $O \in \Gamma_{6t^\circ}$ and control (5.8) is optimal with $t_1 = t^\circ$. It is not difficult to show that the trajectory L° cannot be optimal when $O \in \Gamma_{10} \cap G_1$. Therefore, $O \in G$, which contradicts the theorem's hypothesis. The theorem is proved.

It can be shown that if $O \in \Gamma_{50}$, then during the motion the origin gets first into the region Γ_{6t} and next into Γ_{1t}. The origin cannot get into Γ_{5t} from region Γ_{6t} or into Γ_{6t} from Γ_{1t}. The rule 6.1 permits us to select the control at the initial instant. If a current time t is taken as being initial, we obatin a feedback control law

$$u_1(t) = \operatorname{sign}\sigma_t, \quad u_2(t) = 1, \quad O \in \Gamma_{1t}, \quad \sigma_t \neq 0$$

$$u_1(t) = \operatorname{sign}\sigma_t, \quad u_2(t) = -1, \quad O \in \Gamma_{6t}, \quad \sigma_t \neq 0$$

$$u_1(t) = -\operatorname{sign}\sigma_t, \quad u_2(t) = -1, \quad O \in \Gamma_{5t}, \quad \sigma_t \neq 0$$

$$u_1(t) = 0, \quad u_2(t) = 1, \quad \sigma_t = 0, \quad \sigma_{1t} < 0$$

$$u_1(t) = \pm 1, \quad u_2 = \begin{cases} +1, & 0 \in \Gamma_{1t}, \\ -1, & 0 \in \Gamma_{6t}, \end{cases} \quad \sigma_t = 0, \quad \sigma_{1t} > 0$$

$$\sigma_t = x(t)\sin\varphi(t) - y(t)\cos\varphi(t), \quad \sigma_{1t} = x(t)\cos\varphi(t) + y(t)\sin\varphi(t)$$

Thus, we have synthesized a control from class X. According to Theorem 6.1 this control is optimal when $O \overline{\in} G$.

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